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# Renormalizability of Quantum Gravity near Two Dimensions

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## Abstract

We study the renormalizability of quantum gravity near two dimensions. Our formalism starts with the tree action which is invariant under the volume preserving diffeomorphism. We identify the BRS invariance which originates from the full diffeomorphism invariance. We study the Ward-Takahashi identities to determine the general structure of the counter terms. We prove to all orders that the counter terms can be supplied by the coupling and the wave function renormalization of the tree action. The bare action can be constructed to be the Einstein action form which ensures the full diffeomorphism invariance.

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# 1 Introduction

As it is well known, the renormalizable field theories are those with dimensionless coupling constants. Those theories are also classically conformally invariant. As a nonperturbative definition of the continuum field theory, we may consider the Euclidean statistical systems on the lattice. The continuum field theory is defined in the vicinity of the critical point of the system where it exhibits the conformal invariance.

It is therefore reasonable to postulate that the short distance limit of the consistent field theories exhibits the conformal invariance. This postulate holds nonperturbatively and the nonlinear sigma models between two and four dimensions are such examples. Nevertheless we can devise a perturbative expansion around two dimensions by studying these theories in the  $2 + \epsilon$  dimensions. In this expansion,  $\epsilon$  is regarded to be a small expansion parameter and the theory is weakly coupled at the short distance fixed point of the renormalization group if  $\epsilon$  is small.

The Einstein gravity may fall into this category. It is classically conformally invariant and renormalizable in two dimensions. Furthermore it is topologically invariant in two dimensions. The Newton (gravitational) coupling constant is found to be asymptotically free[1, 2]. The investigations in the context of the string theory and the matrix models have vindicated the asymptotic freedom of the gravitational coupling constant in two dimensions. Therefore it is similar to the nonlinear sigma models and it is tempting to conjecture that the short distance fixed point in the renormalization group exists beyond two dimensions. It implies the existence of the consistent quantum gravity beyond two dimensions.

In our investigation of this problem, the dynamics of the conformal mode of the metric is found to be very different from the rest of the degrees of freedom[3, 4, 5]. Therefore we need to treat these two different variables differently. For this purpose, we decompose the metric into the conformal factor and the rest as  $g_{\mu\nu} = \hat{g}_{\mu\rho}(e^h)^\rho_\nu e^{-\phi} = \tilde{g}_{\mu\nu}e^{-\phi}$ . Here we have also introduced a background metric  $\hat{g}_{\mu\nu}$ . The tensor indices of the fields are raised and lowered by the background metric.  $h_{\mu\nu}$  is a traceless symmetric tensor. The pure Einstein action in

this parametrization is:

$$I_{Einstein} = \int \frac{\mu^\epsilon}{G} e^{-\frac{\epsilon}{2}\phi} \{ \tilde{R} - \frac{\epsilon(D-1)}{4} \tilde{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \}, \quad (1)$$

where  $\int = \int d^D x \sqrt{\tilde{g}}$  denotes the integration over the  $D$  dimensional spacetime.  $\tilde{R}$  is the scalar curvature made out of  $\tilde{g}_{\mu\nu}$ .  $G$  is the gravitational coupling constant and  $\mu$  is the renormalization scale to define it.

The  $\beta$  function of the gravitational coupling constant ( $\beta_G$ ) in  $D = 2 + \epsilon$  dimensions at the one loop level is

$$\mu \frac{\partial}{\partial \mu} G = \epsilon G - \frac{25-c}{24\pi} G^2, \quad (2)$$

where  $c$  counts the matter contents. It shows that the theory is well defined at short distance as long as  $c < 25$ . The short distance fixed point of the  $\beta$  function is  $G^* = 24\pi\epsilon/(25-c)$ .

The pure Einstein action can be rewritten in the following way by the change of the variables with respect to the conformal mode:

$$I_{gravity} = \int \frac{\mu^\epsilon}{G} \{ \tilde{R}(1 + a\psi + \epsilon b\psi^2) - \frac{1}{2} \tilde{g}^{\mu\nu} \partial_\mu \psi \partial_\nu \psi \}. \quad (3)$$

In this expression, the kinetic term for the conformal mode becomes canonical. Classically  $a^2 = 4\epsilon b = \epsilon/2(D-1)$ .

$a^2$  is an indicator of the conformal mode dependence of the theory (conformal anomaly). As it is well known the conformal anomaly is synonymous to the  $\beta$  functions. Therefore the nontrivial  $\beta$  function (2) implies that  $a^2$  will be renormalized at the quantum level. It will be shown that it is related to the  $\beta$  function of  $G$  as  $a^2 G = \beta_G/2(D-1)$  in section 4. Therefore  $a^2$  can be expanded by  $G$  as

$$a^2 = \frac{1}{2(D-1)} (\epsilon - AG - 2BG^2 \dots). \quad (4)$$

We encounter  $1/\epsilon$  poles if we expand  $a$  in terms of  $G$  since  $a^2$  starts with  $\epsilon$ . For this reason we treat  $a$  as another coupling constant of the theory whose square is related to  $\beta_G$  by (4). We start with the tree action which generalizes the Einstein action in this way. We refer the generalized Einstein action in eq. (3) as the gravity action in this paper.

It is straightforward to couple the matter in the conformally invariant way by adding the following matter action.

$$I_{matter} = \int \left\{ \frac{1}{2} \tilde{g}^{\mu\nu} \partial_\mu \varphi_i \partial_\nu \varphi_i - \epsilon b \tilde{R} \varphi_i^2 \right\}, \quad (5)$$

where  $i$  which runs up to  $c$  counts the matter contents. In this parametrization, we have rescaled the matter field by the conformal mode to show the decoupling of the conformal mode explicitly.

Although  $G$  governs the dynamics of  $h_{\mu\nu}$  field, the dynamics of the conformal mode is governed by the coefficient  $a$  which has a singular expansion in  $G$ . This singular expansion arises due to the presence of the kinematical poles[2]. This is the origin of the difficulty to carry out the  $2 + \epsilon$  dimensional expansion of quantum gravity. Our formulation resums the kinematical poles to all orders in  $G$ [3]. For this purpose, we have proposed that we should treat  $G$  and  $a$  as independent couplings. We consider a tree level action which is invariant under the volume preserving diffeomorphism. The general covariance can be recovered by further imposing the conformal invariance on the theory with respect to the background metric[4, 5]. This requirement certainly determines the relation between  $G$  and  $a$  at the classical level. At the quantum level, this relation receives corrections due to the  $\beta$  function just like eq. (4).

Concerning this background independence requirement, we realize that the Einstein action is certainly a solution for it. It is in fact to be a unique one. Therefore we conclude that the bare action which is obtained by adding the counter terms to the tree level action is the Einstein action. This point has been found to be the case at the one loop level calculation[5]. In our formulation, the tree action captures the dominant dynamics whose corrections are small in the perturbation theory. The question now is whether we can renormalize the theory in this scheme to all orders in the perturbation theory.

In [7], one of us has proposed such a proof based on the Ward-Takahashi identity. This identity follows from the gauge invariance of the theory just like in generic gauge theories. However in that work, the identity is assumed to be valid up to only the required orders in the perturbation theory. In this work we require that the WT identity to be the exact

identity on the theory. Such a requirement is very restrictive. It leaves us no choice except to conclude that the bare action is the Einstein action. However we still assume that the tree level action is invariant under the volume preserving diffeomorphism only. This is because we need to treat the dynamics of  $h_{\mu\nu}$  and  $\psi$  fields differently. In this paper we construct a proof which shows that the theory is renormalizable to all orders within this scheme. We have thus further clarified the structure of the bare action and how the general covariance is ensured in our formalism.

We further study the renormalization of the relevant operators such as the cosmological constant operator. This question is certainly crucial for the theory to be physically meaningful. We prove that the cosmological constant operator is multiplicatively renormalizable. Just like in two dimensions, the anomalous dimensions are  $O(1)$  in general and which forces us particular considerations. However the anomalous dimensions are calculable near two dimensions by the saddle point method. Although the cosmological constant operator is multiplicatively renormalizable, it may be useful to consider the renormalized cosmological constant operator which incorporates the quantum effect. The functional form of the renormalized cosmological constant operator is fixed by requiring that the renormalized cosmological constant operator is background independent. We argue that the bare cosmological constant operator which is obtained by adding the necessary counter terms to the renormalized cosmological constant operator is again of the generally covariant form.

The organization of this paper is as follows. In section two, we set up the BRS formalism and derive the exact Ward-Takahashi identities. In section three, we solve the WT identity to determine the bare action. We give the inductive proof of the renormalizability in section four. We show that the divergences of the theory can be canceled by the counter terms which can be supplied by the coupling constant and the wave function renormalization of the tree action. We also show that the bare action can be constructed to be the Einstein action form. In section five, we study the renormalization of the cosmological constant operator. We discuss a physical definition of the  $\beta$  function in quantum gravity. We conclude in section six with discussions.

## 2 BRS Invariance and Ward-Takahashi Identity

In this section, we set up the BRS formalism in quantum gravity and derive the exact Ward-Takahashi identities. These identities are the consequence of the general covariance.

We adopt the action (3) as the tree level action. The coefficient  $a$  appears with the single power of  $\psi$ . Therefore there is a parity invariance under the simultaneous change of the signs of  $a$  and  $\psi$ . We can classify the effective action into the even and odd parity sectors.  $\psi$  field appears as the even and odd powers in each sector. Due to this parity invariance, only  $a^2$  appears in the quantum corrections in the even parity sector. In the odd parity sector, the situation is the same apart from the over all factor of  $a$ . Therefore these corrections can be expanded in terms of  $G$  by using the relation of (4). These arguments hold as long as we choose the gauge fixing terms which also respect the parity invariance.

The crucial symmetry of the theory is the invariance under the diffeomorphism. The metric changes under the general coordinate transformation as:

$$\delta g_{\mu\nu} = \partial_\mu \epsilon^\rho g_{\rho\nu} + g_{\mu\rho} \partial_\nu \epsilon^\rho + \epsilon^\rho \partial_\rho g_{\mu\nu}. \quad (6)$$

We have decomposed the metric into the conformal mode and the rest as  $g_{\mu\nu} = \tilde{g}_{\mu\nu} \psi^{\frac{4}{\epsilon}}$ . The over all scale of  $\psi$  is irrelevant in this paragraph. Here  $\det \tilde{g} = \det \hat{g}$  since  $\tilde{g} = \hat{g} e^h$ . In this decomposition, the general coordinate transformation takes the following form:

$$\begin{aligned} \delta \tilde{g}_{\mu\nu} &= \partial_\mu \epsilon^\rho \tilde{g}_{\rho\nu} + \tilde{g}_{\mu\rho} \partial_\nu \epsilon^\rho + \epsilon^\rho \partial_\rho \tilde{g}_{\mu\nu} - \frac{2}{D} \nabla_\rho \epsilon^\rho \tilde{g}_{\mu\nu}, \\ \delta \psi &= \epsilon^\rho \partial_\rho \psi + ((D-1)a + \frac{\epsilon}{4} \psi) \frac{2}{D} \nabla_\rho \epsilon^\rho, \\ \delta \varphi_i &= \epsilon^\rho \partial_\rho \varphi_i + (\frac{\epsilon}{4} \varphi_i) \frac{2}{D} \nabla_\rho \epsilon^\rho, \end{aligned} \quad (7)$$

where the covariant derivative is taken with respect to the background metric. An arbitrary constant  $a$  can be introduced here by the constant shift of  $\psi$ , although this shift is singular in  $\epsilon$ .  $a$  may be viewed as the vacuum expectation value of  $\psi$ . The matter fields transform as above since we have scaled the matter fields by a single factor of  $\psi$  in eq. (5).

As we have explained, the odd parity sector has the single power of  $a$  as the overall factor and the even powers of  $a$  always appear apart from this overall factor. (7) is consistent with

such a structure since the single power of  $a$  appears when the powers of  $\psi$  are reduced by one. Therefore this invariance can be enforced on the theory by using the relation (4) without expanding  $a$  by  $G$ .

In order to prove the renormalizability of the theory, we set up the BRS formalism[8]. The BRS transformation of these fields  $\delta_B$  is defined by replacing the gauge parameter by the ghost field  $\epsilon^\mu \rightarrow C^\mu$ . The BRS transformation of  $h_{\mu\nu}$  field is defined through the relation  $\tilde{g} = \hat{g}e^h$ . The BRS transformation of ghost, antighost and auxiliary field is

$$\begin{aligned}\delta_B C^\mu &= C^\nu \nabla_\nu C^\mu, \\ \delta_B \bar{C}^\mu &= \lambda^\mu, \\ \delta_B \lambda^\mu &= 0.\end{aligned}\tag{8}$$

The BRS transformation can be shown to be nilpotent  $\delta_B^2 = 0$ .

We denote  $A_i = (h_{\mu\nu}, \psi, \varphi_j)$ . We also introduce a gauge fixing function  $F_\alpha(A_i)$ . It is an arbitrary function of  $A_i$  with dimension one. We assume that it respects the parity invariance of the tree action. The gauge fixed action is

$$S = I + \frac{\mu^\epsilon}{G} \int [-\lambda_\alpha \lambda^\alpha + \lambda^\alpha F_\alpha - \bar{C}^\alpha \delta_B F_\alpha - K^i \delta_B A_i - L_\alpha \delta_B C^\alpha].\tag{9}$$

Here we have introduced sources  $K$  and  $L$  for the composite operators.  $I = I_{\text{gravity}} + I_{\text{matter}}$  is the total action without the BRS exact terms. In what follows,  $I$  will be referred as the tree action.

The partition function is

$$Z = e^W = \int [dA dC d\bar{C} d\lambda] \exp(-S + \frac{\mu^\epsilon}{G} \int [J^i A_i + \bar{\eta}_\alpha C^\alpha + \bar{C}^\alpha \eta_\alpha + l_\alpha \lambda^\alpha]).\tag{10}$$

By the change of the variables with the BRS transformation form, we obtain the Ward-Takahashi identity for the generating functional of the connected Green's functions:

$$\int (J^i \frac{\delta}{\delta K^i} - \bar{\eta}_\alpha \frac{\delta}{\delta L_\alpha} + \eta_\alpha \frac{\delta}{\delta l_\alpha}) W = 0.\tag{11}$$

The WT identity for the effective action is obtained by the Legendre transformation:

$$\int [\frac{\delta \Gamma}{\delta A_i} \frac{\delta \Gamma}{\delta K^i} + \frac{\delta \Gamma}{\delta C^\alpha} \frac{\delta \Gamma}{\delta L_\alpha} - \frac{\mu^\epsilon}{G} \lambda^\alpha \frac{\delta \Gamma}{\delta \bar{C}^\alpha}] = 0.\tag{12}$$

In order to make the above expression finite, we need to add all necessary counter terms to  $S$ . The bare action  $S^0$  obtained in this way satisfies the same equation:

$$\int \left[ \frac{\delta S^0}{\delta A_i} \frac{\delta S^0}{\delta K^i} + \frac{\delta S^0}{\delta C^\alpha} \frac{\delta S^0}{\delta L_\alpha} - \frac{\mu^\epsilon}{G} \lambda^\alpha \frac{\delta S^0}{\delta \bar{C}^\alpha} \right] = 0. \quad (13)$$

On the other hand, eq. (11) follows from eq. (13) in dimensional regularization. To simplify notations, we introduce an auxiliary field  $M_\alpha$  and add to the action the combination  $-\frac{\mu^\epsilon}{G} \int M_\alpha \lambda^\alpha$  in such a way that  $\frac{\mu^\epsilon}{G} \lambda^\alpha = -\frac{\delta \Gamma}{\delta M_\alpha} = -\frac{\delta S}{\delta M_\alpha}$ . Then the left hand side of eq. (12) and eq. (13) become homogeneous quadratic equations which we write symbolically as  $\Gamma * \Gamma$  and  $S^0 * S^0$ .

In our derivation of the WT identities, we have assumed the invariance of the bare action under the gauge transformation (7). However we start with the tree level action which possesses only the volume preserving diffeomorphism invariance. The crucial question is whether we can choose the counter terms of the theory in such a way to satisfy these identities by starting with such a tree action. We answer affirmatively to this question in this paper.

### 3 Analysis of the Bare Action

In this section, we solve eq. (13) to determine  $S^0$ .  $S^0$  will be simply denoted by  $S$  in this section. Let us examine the general structure of the bare action. By power counting, it has to be a local functional of fields and sources with the dimension  $D$ . We also have the ghost number conservation rule and its ghost number has to be zero. By these dimension and ghost number considerations, it is easy to see that  $K$  and  $L$  appear only linearly in  $S$ :

$$S = \int \frac{1}{G^0} [-K^i (\delta'_B A_i) - L_\alpha (\delta'_B C^\alpha)] + \tilde{S}, \quad (14)$$

where  $\delta'_B$  denotes most general BRS like transformations with the correct dimension and ghost number. It is also easy to see that there are no  $\lambda$  and hence no  $\bar{C}$  dependence in  $\delta'_B$ . Since  $\lambda$  has dimension 1,  $\tilde{S}$  can be at most quadratic in  $\lambda$ :

$$\tilde{S} = \int \frac{1}{G^0} \left[ -\frac{1}{2} \tilde{E}_{\alpha\beta} \lambda^\alpha \lambda^\beta + \lambda^\alpha \tilde{F}_\alpha + \tilde{L} \right], \quad (15)$$

where  $\tilde{E}_{\alpha\beta}$  and  $\tilde{F}_\alpha$  are general local functions of  $A, C$  and  $\bar{C}$  with dimension zero and one respectively.  $G^0$  is the bare gravitational coupling constant and it is the only quantity with dimension  $-\epsilon$ .

We denote below by  $\theta^i$  the set of all anticommuting fields  $K^i, C^\alpha, \bar{C}^\alpha$  and  $x_i$  all commuting fields  $A_i, L_\alpha, M_\alpha$ . The fundamental equation for the action  $S$  takes then the form:

$$\frac{\partial S}{\partial x_i} \frac{\partial S}{\partial \theta_i} = 0. \quad (16)$$

The equation (16) is invariant under the following canonical transformations. Let us make the change of the variables  $(\theta, x) \rightarrow (\theta', x')$ :

$$\begin{aligned} x_i &= \frac{\partial \varphi}{\partial \theta_i}(x', \theta), \\ \theta'_i &= \frac{\partial \varphi}{\partial x'_i}(x', \theta). \end{aligned} \quad (17)$$

We can verify that we recover the equation (16) in the new variables. It has been shown that the set of the canonical transformations (17) is the most general set of the transformations which leaves the equation (16) invariant[8, 12].

Let us write these transformations in the infinitesimal form:

$$\begin{aligned} x'_i &= x_i - \frac{\partial \varphi}{\partial \theta^i}, \\ (\theta^i)' &= \theta^i + \frac{\partial \varphi}{\partial x_i}, \end{aligned} \quad (18)$$

the action  $S(\theta^i, x_i)$  changes as

$$S(\theta', x') - S(\theta, x) = \Delta\varphi, \quad (19)$$

where

$$\Delta = \frac{\partial S}{\partial \theta_i} \frac{\partial}{\partial x_i} + \frac{\partial S}{\partial x_i} \frac{\partial}{\partial \theta_i}. \quad (20)$$

We also have the following relation:

$$\Delta^2 = \left[ -\frac{\partial}{\partial \theta^j} (S * S) \right] \frac{\partial}{\partial x_j} + \left[ \frac{\partial}{\partial x_j} (S * S) \right] \frac{\partial}{\partial \theta^j}. \quad (21)$$

Therefore if  $S$  is the solution of the equation (16),  $S + \Delta\varphi$  is also the solution of it since this is an infinitesimal canonical transformation of the fields. We call  $\Delta\varphi$  a BRS exact solution of the equation (16).

Since we are studying the Einstein gravity, the Einstein action is the only generally covariant action with the dimension  $D$ . The action (9) with the Einstein action for  $I$  certainly satisfies the equation (16). However the solution is not unique due to the freedom in association with the canonical transformation of the fields. On the other hand it is the only freedom of the solutions of (16). Therefore the equations of (14) and (15) have to be interpreted by the canonical transformations. Physically the canonical transformations correspond to the freedom in association with the wave function renormalization and the gauge fixing procedure.

## 4 Inductive Proof of the Renormalizability

In this section, we construct an inductive proof of the renormalizability of quantum gravity by the  $2 + \epsilon$  dimensional expansion approach.

Our analysis is based on the expansion of the effective action by the gravitational coupling constant  $G$ :

$$\Gamma = \sum_{l=0}^{\infty} \Gamma_l, \quad (22)$$

in which  $\Gamma_0$  is the tree level action  $S$ . We define  $\Gamma_l$  to be the effective action of  $G^{l-1}$  order. Our formalism contains two dimensionless parameters  $G$  and  $a$ . Although  $a^2$  possesses the expansion by  $G$  as in (4), it starts with the quantity of  $O(\epsilon)$  and the expansion of  $a$  by  $G$  is singular. Therefore the effective action can be expanded by  $G$  apart from the overall factor of  $a$  in the odd parity sector.  $a$  is regarded as  $(G)^0$  and the expansion of the effective action by  $G$  should be understood in this sense.

Hence the effective action  $\Gamma$  and the bare action  $S_0$  consist of the even and odd sectors as

$$\Gamma = \Gamma_{even} + a\Gamma_{odd},$$

$$S^0 = S_{even}^0 + aS_{odd}^0, \quad (23)$$

where we have written  $a$  dependence explicitly.  $\Gamma_{even(odd)}$  and  $S_{even(odd)}^0$  can be expanded in  $G$  alone by using eq. (4). What we would like to prove is that we can choose  $S^0$  which makes  $\Gamma$  finite in such a way that  $\Gamma * \Gamma = 0$ . In dimensional regularization, the bare action  $S^0$  also satisfies  $S^0 * S^0 = 0$ .

In order to determine the effective action at  $G^{l-1}$  order, the  $l$  loop level computation is required.  $a^2$  is also determined up to  $G^l$  order by this computation.  $\Gamma_l$  differs from the conventional  $l$  loop level effective action since  $a^2$  can be expanded in  $G$ . Hence it also receives contributions from the lower loop level.

We assume as an induction hypothesis that we have been able to construct the bare action  $S_{l-1}^0$  which satisfies  $S_{l-1}^0 * S_{l-1}^0 = 0$  and renders  $\Gamma$  finite up to  $G^{l-2}$  order by the  $l-1$  loop level computation. Namely  $\Gamma_k$  with  $k \leq l-1$  are assumed to be finite. We denote  $a$  which appears in  $S_{l-1}^0$  as  $a_{l-1}$  and it is regarded as  $(G)^0$ .  $S_{l-1}^0$  consists of the even and the odd parity sectors and the odd parity sector is multiplied by  $a_{l-1}$ . The situation is the same with the effective action  $\Gamma$  and the both  $\Gamma_{even}$  and  $\Gamma_{odd}$  are assumed to be finite up to  $G^{l-2}$  order.  $a_{l-1}^2$  is assumed to be determined up to order  $G^{l-1}$ :

$$a_{l-1}^2 = \frac{1}{2(D-1)}(\epsilon - AG \dots - \lambda_{l-1}^1 G^{l-1}). \quad (24)$$

Although the bare action is taken to satisfy  $S^0 * S^0 = 0$ , we have adopted the tree level action  $S$  in such a way that  $S * S$  is of higher orders in  $G$ . This choice is motivated by the presence of the conformal anomaly in quantum gravity. By starting with such a tree action, our formalism can handle the dynamics of the conformal mode which is influenced by the conformal anomaly. From our basic equation  $\Gamma * \Gamma = 0$ , we find the following relation at  $G^{l-2}$  order:

$$S * \Gamma_l + \Gamma_l * S = \Delta \Gamma_l = - \sum_{k=0}^{l-1} \sum_{m=0}^k \Gamma_m * \Gamma_{k-m}. \quad (25)$$

We recall that  $\Gamma_l$  is  $O(G^{l-1})$ . The right hand side of this equation has to be at least  $O(G^{l-2})$  by the inductive assumption and we only consider the quantities of  $G^{l-2}$  order in this equation. By the induction hypothesis, the right hand side of this equation is finite.

The reason is that it involves only  $\Gamma_k$  with  $k \leq l-1$ . If  $a_{l-1}^2$  is obtained in this equation, we expand it by  $G$ . This expansion terminates at order  $G^{l-1}$  by the inductive assumption. Obviously we find no divergence by doing that on the right hand side. On the other hand, the left hand side contains  $\Gamma_l$  which is divergent in general. This equation therefore determines the possible structure of the divergent part of  $\Gamma_l^{div}$  at  $G^{l-1}$  order.

In the perturbative expansion of the field theory, all divergences at the  $l$  loop order are guaranteed to be local as long as all subdiagrams are subtracted to be finite. It is because such divergences can be made finite by differentiating the external momenta. We have assumed by the induction hypothesis that the effective action has been made to be finite up to  $G^{l-2}$  order. At  $G^{l-1}$  order, all subdiagrams are at most  $G^{l-2}$  order. We can then conclude that all divergences at  $G^{l-1}$  order are local by using the above argument. Here we would like to discuss the treatment of the divergences of  $a^2/\epsilon$  type. In the leading order, they can be regarded as finite. In the minimal subtraction scheme, we need not subtract them. However we find divergences if we expand them by  $G$  to higher orders. We need to subtract them even in the minimal subtraction scheme in higher orders. Another possibility is to subtract them from the leading order as a whole. Such a subtraction scheme might have some advantage in our formalism since we can subtract the class of terms in consideration at once.

We have thus reduced the question of the renormalizability to that of finding the most general solutions of eq. (25). The similar problems have been investigated extensively in gauge theories[8, 12]. We can find the following solutions of this equation based on the results of such investigations.

Let  $\varphi$  be a local functional of fields at  $G^l$  order. Then  $\Delta\varphi$  is a local functional of fields at  $G^{l-1}$  order. The divergence of this form is consistent with eq. (25) since  $\Delta^2\varphi = 0$  at  $G^{l-2}$  order. This type of the divergence is called as the BRS exact part.  $\Gamma_i^{div}$  can be decomposed into the BRS exact part and the rest in general. We call the rest of the divergence as the nontrivial solution of eq. (25).

The divergences of the tree action form eq. (3) and eq. (5) which are of order  $G^{l-1}$  do satisfy this equation. It is because the tree action is generally covariant to the leading

order. However this equation allows more general classes of the divergences which can be seen as follows. Let us consider a generic local action which is invariant under the volume preserving diffeomorphism. It is easy to see that such an action is invariant under the gauge transformation eq. (7) if it is invariant under the following conformal transformation:

$$\begin{aligned}\delta\psi &= (D-1)(a+2\epsilon b\psi)\delta\bar{\phi}, \\ \delta\varphi_i &= 2\epsilon b(D-1)\varphi_i\delta\bar{\phi}, \\ \delta\hat{g}_{\mu\nu} &= -\hat{g}_{\mu\nu}\delta\bar{\phi}.\end{aligned}\tag{26}$$

When we plug such an action as  $\Gamma_l$  on the left hand side of eq. (25), we find that it is proportional to the conformal anomaly of the action. The conformal anomaly vanishes in the two dimensional limit if the action becomes conformally invariant in two dimensions. The simple pole divergences of this type thus result in the finite conformal anomaly. Therefore such divergences are consistent with eq. (25).

The nontrivial divergences can be classified into the two types: those with the simple pole in  $\epsilon$  and those with higher poles in  $\epsilon$ . From the considerations we have just gone through, we find that the higher pole divergences of the tree action form are consistent with our basic equation. The simple pole divergences which are invariant under the volume preserving diffeomorphism are also consistent with eq. (25) if they are conformally invariant in the two dimensional limit. It is because  $\Delta\Gamma_l^{div}$  is given by the finite conformal anomaly for such divergences.

Through these considerations, we have found very general solutions for the possible divergences which are consistent with eq. (25). We have classified them into the BRS exact and the nontrivial solutions. The nontrivial solutions are classified into the two different types. They are those of the tree action type and those with the finite conformal anomaly type. It is physically very plausible that they are the only solutions of this equation. As the major conjecture in this proof we assume that the only solutions of eq. (25) are those we have found in this section.

In this model, the only operator which is invariant under the volume preserving diffeomorphism and which is conformally invariant in two dimensions is  $\int \tilde{R}$ . This operator is

invariant under the transformation eq. (7) modulo  $O(\epsilon)$ . We adopt this operator and the gravity and the matter actions as the independent operators.  $\psi$  field transforms in a specific way in this gauge transformation. The conformal transformation eq. (26) is the specific type which is a part of the gauge transformation. We remark that the operator  $\int \tilde{g}^{\mu\nu} \partial_\mu \psi \partial_\nu \psi$  is not invariant under eq. (26) in two dimensional limit in our sense since we regard  $a$  as a finite coupling constant.  $\int \tilde{g}^{\mu\nu} \partial_\mu \varphi_i \partial_\nu \varphi_i$  will be regarded as the matter action to the leading order of  $\epsilon$ .

Therefore at  $l$  loop level, new divergences of the following form may arise:

$$\frac{\mu^\epsilon}{G} \int \left[ \frac{\lambda_l^1 G^l}{\epsilon} \tilde{R} \right] + \text{tree action.} \quad (27)$$

Here we denote the residue of the simple pole at the  $l$  loop level in association with  $\int \tilde{R}$  by  $\lambda_l^1$ . The divergences of the gravity action form can be subtracted by the renormalization of the gravitational coupling constant  $G$ . The divergences of the matter action type can be subtracted by the wave function renormalization of  $\varphi_i$ . We assume here that the gauge fixing function  $F_\alpha(A_i)$  does not depend on the matter fields for simplicity. In order to cancel the remaining divergence, we need the counter term  $-\mu^\epsilon \int (\lambda_l^1 G^{l-1}/\epsilon) \tilde{R}$ . The bare action constructed in this way have to be the Einstein action form in order to satisfy  $S^0 * S^0 = 0$  as we have found in the previous section. The pure Einstein action in the parametrization we have adopted is

$$\frac{\mu^\epsilon}{G} \int \left[ \tilde{R} \left( \frac{2(D-1)a^2}{\epsilon} + a\psi + \epsilon b\psi^2 \right) - \frac{1}{2} \partial_\mu \psi \partial_\nu \psi \tilde{g}^{\mu\nu} \right]. \quad (28)$$

Therefore  $G^l$  order part of  $2(D-1)a^2$  is determined from this requirement to be  $-\lambda_l^1 G^l$ . We point out that  $\lambda_l^1$  itself does not depend on the  $G^l$  order part of  $2(D-1)a^2$  since it comes from the quantum loop effect.

We still need to study the BRS exact divergences which can be expressed as  $\Delta\varphi$ . The general form of  $\varphi$  is:

$$\varphi = \int [K^i \Psi'_i + L_\alpha \Theta'^\alpha_\beta C^\beta + \bar{C}^\alpha (F'_\alpha + \lambda^\beta E'_{\alpha\beta})], \quad (29)$$

where  $\Psi'$  and  $\Theta'$  are general local functions of  $A, C, \bar{C}$  with dimension zero and vanishing ghost number. As we have explained, the BRS exact part can be associated with a canonical transformation on the fields. Here we consider the physical implications of these canonical transformations. Under this transformation, the part of  $S^0$  linear in  $K$  and  $L$  changes as:

$$\begin{aligned} K^i \delta_B A_i &\rightarrow K^i \delta_B A_i + K^i \delta_B (\Psi'_i) - K^i \frac{\partial \delta_B A_i}{\partial A_j} \Psi'_j + K^i \frac{\partial \delta_B A_i}{\partial C^\alpha} \Theta'_{\beta}^{\alpha} C^\beta, \\ L_\alpha \delta_B C^\alpha &\rightarrow L_\alpha \delta_B C^\alpha - L_\alpha \delta_B (\Theta'_{\beta}^{\alpha} C^\beta) + L_\alpha \frac{\partial \delta_B C^\alpha}{\partial C^\beta} \Theta'_{\gamma}^{\beta} C^\gamma - L_\alpha \frac{\partial \delta_B C^\alpha}{\partial A_i} \Psi'_i. \end{aligned} \quad (30)$$

These infinitesimal deformations can be interpreted as the change of the functional form of the BRS transformation in association with the wave function renormalization of the fields. Note that the functional form of the BRS transformation has to change in terms of the renormalized variables, although the functional form of the BRS transformation remains the same in terms of the bare fields. The renormalized BRS transformation continues to be nilpotent. The rest of the BRS exact part causes the renormalization of the gauge fixing part.

By defining the bare action at  $l$  loop level

$$S_l^0 = S_{l-1}^0 - \Gamma_l^{div} + \text{higher orders}, \quad (31)$$

it is possible to render  $\Gamma$  finite up to order  $G^{l-1}$ . The BRS exact counter terms can be interpreted as the renormalization of the wave functions and the gauge fixing part. The rest of the counter terms can be interpreted as the coupling constant renormalization of the tree level action. The higher order terms in eq. (31) has to be chosen in such a way that  $S_l^0$  satisfies  $S_l^0 * S_l^0 = 0$  exactly. As we have explained in the previous section, such a bare action has to be the gauge fixed Einstein form modulo the canonical transformation of the fields. Since  $\Gamma_l^{div}$  is also of this type, we can construct such a bare action by integrating these infinitesimal deformations.  $a^2$  is now determined up to order  $G^l$ .

When we obtain  $S_l^0$  from  $S_{l-1}^0$ , we substitute  $a_l$  for  $a_{l-1}$ . The only difference which has been brought about by this change is the addition of the required counter term of  $\int \tilde{R}$  type at order  $G^{l-1}$  apart from the change of the definition of  $a$  up to order  $G^{l-1}$ . Now the circle is

complete and we have proven the renormalizability of quantum gravity near two dimensions. The major assumption we have made in this proof is that the solutions we have found in this section exhaust the solutions of eq. (25).

Under this very plausible assumption, we have established that this model is renormalizable to all orders with the following bare action:

$$\frac{\mu^\epsilon Z}{G} \int [\tilde{R}(Z_G + a\psi + \epsilon b\psi^2) - \frac{1}{2}\partial_\mu\psi\partial_\nu\psi\tilde{g}^{\mu\nu}] + \int [\frac{1}{2}\partial_\mu\varphi_i\partial_\nu\varphi_i\tilde{g}^{\mu\nu} - \epsilon b\tilde{R}\varphi_i^2], \quad (32)$$

where  $Z_G = 1 - \lambda^1/\epsilon$  and  $2(D - 1)a^2 = \epsilon - \lambda^1$ . Here we have omitted the BRS exact part of the action. Of course we also need counter terms which correspond to the wave function renormalization of the action.

Let us consider the physical significance of the coefficient  $a^2$ . We note that it measures the conformal anomaly of the theory. At the critical point where  $a$  vanishes, the conformal mode becomes indistinguishable from the scalar fields which couple to the gravity in the conformally invariant way. The  $Z_2$  invariance under  $\psi \rightarrow -\psi$  is restored at the critical point since the odd parity sector of the effective action vanishes. We have suggested that this  $Z_2$  invariance may distinguish the different phases of quantum gravity[5].

We would like to interpret this phenomenon as the signature of the conformal invariance.  $a^2$  can be expanded in  $G$  as follows:

$$2(D - 1)a^2G = \epsilon G - AG^2 - 2BG^3 \dots \quad (33)$$

This quantity measures the conformal anomaly of the theory as a function of  $G$ . For the particular value of  $G$ , it vanishes and the theory becomes conformally invariant in the sense we have just explained. As it is well known the  $\beta$  function is related to the conformal anomaly. Therefore it is reasonable to adopt (33) as the  $\beta$  function of  $G$  by resorting to this connection. We also recall that  $G$  is the gravitational coupling constant. It measures the strength of the coupling of  $h_{\mu\nu}$  field at the momentum scale  $\mu$ .

In the conventional definition of the  $\beta$  function, it is defined through the bare gravitational coupling constant:

$$\frac{1}{G^0} = \frac{\mu^\epsilon}{G} Z'_G. \quad (34)$$

The  $\beta$  function of  $G$  is obtained by demanding that the bare quantity is independent of the renormalization scale  $\mu$ . However there is an ambiguity in this procedure since the bare gravitational coupling constant changes if we rescale the conformal mode[2]. The relation (33) is free from such an ambiguity since this ambiguity does not alter the classical relation.

Therefore we adopt the right hand side of eq. (33) as the  $\beta$  function of  $G$ . We expect that this  $\beta$  function is also obtained by the conventional procedure since  $Z'_G = ZZ_G$  in our scheme. Through the conventional procedure we find that  $\mu \frac{\partial}{\partial \mu} G = \epsilon G Z'_G / (1 - G \frac{\partial}{\partial G}) Z'_G$ . If  $Z = (1 - G \frac{\partial}{\partial G}) Z'_G$ , we find the same  $\beta$  function with eq. (33) by the conventional method. In our renormalization procedure, we have classified the nontrivial solutions of eq. (25) into those with the conformal anomaly and those with the vanishing conformal anomaly. The latter is associated with the higher poles in  $\epsilon$  in general while the former is associated with the simple pole. This classification must be generic in field theories. We expect that such a classification of divergences underlies the pole identities which ensure the finiteness of the conformal anomaly.

Before concluding this section, we evaluate the conformal anomaly of the bare action with respect to the background metric. Under the conformal transformation eq. (26), the bare action eq. (32) changes as:

$$\frac{\mu^\epsilon}{2G} \int (\epsilon Z_G - 2(D-1)a^2) \tilde{R}_r \delta \bar{\phi}, \quad (35)$$

where  $\tilde{R}_r = Z \tilde{R}$  is the renormalized operator. The coefficient of  $\tilde{R}_r$  is called as a trace anomaly coefficient. It is expressed in terms of the  $\beta$  functions since  $\beta_G = \epsilon Z_G G$ . We observe that it certainly vanishes by the construction. We have proposed to construct the quantum gravity by requiring that it does not depend on the conformal mode of the background metric[4, 5, 7]. We have shown how this requirement is fulfilled in our renormalization scheme.

## 5 Renormalization of the cosmological constant operator

In this section, we study the renormalization of the cosmological constant operator. The renormalization of the relevant spinless operators can be done in the analogous procedures. This problem has been studied extensively in our previous works[3, 4, 5]. We prove that the cosmological constant operator is multiplicatively renormalizable by using the WT identity. However the quantum corrections are  $O(1)$  in general. It may be useful to define the renormalized cosmological constant operator which incorporates the large quantum renormalization effect. We can determine the functional form of the renormalized cosmological constant operator by requiring the background metric independence. We further make the relation between the bare and the renormalized cosmological constant operators explicit in this section.

The bare cosmological constant operator is of the generally covariant form:

$$\begin{aligned} \int d^D x \sqrt{g} &= \int \left(1 + \frac{\epsilon b}{a} \psi\right)^{\frac{2D}{\epsilon}} \\ &\sim \int \exp\left(\frac{1}{a} \psi - \frac{\epsilon}{8a^2} \psi^2 \dots\right). \end{aligned} \quad (36)$$

This operator is invariant under the gauge transformation eq. (7) for an arbitrary value of  $a$ . We consider the infinitesimal perturbation of the theory by the cosmological constant operator. The new tree action is  $S + \Lambda \tilde{S}$  where  $\tilde{S}$  is the cosmological constant operator. The effective action can also be expanded in  $\Lambda$  as  $\Gamma + \Lambda \tilde{\Gamma}$ . The WT identity eq. (12) becomes at  $O(\Lambda)$  as:

$$\Gamma * \tilde{\Gamma} + \tilde{\Gamma} * \Gamma = 0. \quad (37)$$

The new tree action satisfies this WT identity. We consider the perturbative expansion of the theory by  $G$ . The inverse powers of  $a$  appear in the cosmological constant operator. We regard the inverse powers of  $a$  as  $O(1)$  in this paragraph. Let us assume by the induction hypothesis that we have made the effective action finite up to  $G^{l-2}$  order. The effective

action at  $G^{l-1}$  order satisfies the following WT identity:

$$S * \tilde{\Gamma}_l + \tilde{\Gamma}_l * S + \tilde{S} * \Gamma_l + \Gamma_l * \tilde{S} = \sum_{k=1}^{l-1} (\Gamma_k * \tilde{\Gamma}_{l-k} + \tilde{\Gamma}_{l-k} * \Gamma_k), \quad (38)$$

where we have also expanded  $\tilde{\Gamma}$  by  $G$ . This equation can determine the general structure of the counter terms of dimension zero which are required to cancel the divergent part of  $\tilde{\Gamma}_l$ .

We find that the divergences of the bare cosmological constant operator form is consistent with this equation. Unlike the dimension two operators, we cannot find other nontrivial solutions. We conjecture that the only BRS nontrivial divergences of dimension zero take the bare cosmological constant operator form. Hence the cosmological constant operator is multiplicatively renormalizable. However the anomalous dimensions are not small even in the perturbation theory since it is  $O(\frac{G}{a^2})$  and  $a^2$  is  $O(G)$  at short distance. Therefore we need to sum up  $O(1)$  quantities in this counting to all orders.

Such a resummation of the leading contributions to all orders has been done by the following method. The propagator for the conformal mode can be read off from the action eq. (3) for small  $\epsilon$  and  $G$  as:

$$\langle \psi(p) \psi(-p) \rangle = \frac{G}{p^2} e^{\frac{\epsilon}{2} \bar{\phi}}, \quad (39)$$

where we have adopted the gauge fixing term which eliminates the mixing between  $h_{\mu\nu}$  and  $\psi$  fields. We have also shown the dependence on the background conformal factor ( $e^{-\bar{\phi}}$ ) explicitly. The divergent part of the vacuum expectation value of the square of the  $\psi$  field is:

$$\langle \psi^2 \rangle = \frac{1}{\pi \epsilon} \frac{G}{2} e^{\frac{\epsilon}{2} \bar{\phi}}. \quad (40)$$

We evaluate the anomalous dimension of this operator by using the propagator eq. (39) for the conformal mode. In order to do so, we may utilize a zero dimensional model with the following action which reproduces eq. (40):

$$\frac{a^2}{G} \pi \epsilon \psi^2 e^{-\frac{\epsilon}{2} \bar{\phi}}. \quad (41)$$

Then the vacuum expectation value of the cosmological constant operator is:

$$\langle \sqrt{g} \rangle = \int d\psi e^{-\frac{D}{2} \bar{\phi}} \exp\left(\frac{4}{\epsilon} \log\left(1 + \frac{\epsilon}{4a} \psi\right) - \frac{1}{G} \pi \epsilon \psi^2 e^{-\frac{\epsilon}{2} \bar{\phi}}\right). \quad (42)$$

This integral can be evaluated exactly for small  $\epsilon$  by the saddle point method after scaling the integration variable  $\psi$  by  $\epsilon$ .

In this way the divergent part of the integral is found to be:

$$\exp\left(\frac{4}{\epsilon}\log(1+\rho_0) - \frac{16a^2\pi}{G\epsilon}\rho_0^2 e^{-\frac{\epsilon}{2}\bar{\phi}}\right), \quad (43)$$

where

$$\rho_0 = \frac{1}{2}\left(-1 + \sqrt{1 + \frac{G}{2a^2\pi}}\right). \quad (44)$$

The anomalous dimension is found by inspecting the  $\bar{\phi}$  dependence of this result to be

$$\gamma = \rho_0^2 \frac{16a^2\pi}{G} = 2 - \frac{16a^2\pi}{G}\rho_0. \quad (45)$$

By combining the canonical dimension of the cosmological constant operator which is the classical  $\bar{\phi}$  dependence of it, the scaling dimension of this operator is found to be  $\frac{16a^2\pi}{G}\rho_0$  to the leading order in  $\epsilon$ . At the short distance fixed point, it behaves as  $a\sqrt{\frac{32\pi}{G}}$ .

Since our results in this section such as (44) involve the inverse power of  $a^2$ , it is crucial to regard  $a^2$  as an independent and a finite coupling. Although it is  $O(\epsilon)$  classically, it receives quantum corrections of  $O(G)$ . In fact we have assumed that it is as small as  $G$  and resummed  $O(1)$  quantities in such a counting to all orders by the saddle point method.

We now consider a physical  $\beta$  function of the theory. One of the physical definition of the  $\beta$  function is to compare the gravitational coupling constant  $G$  and the cosmological constant  $\Lambda$  [2, 11]. If we divide eq. (33) by the scaling dimension of the cosmological constant operator, we obtain  $\frac{aG^2}{\sqrt{8\pi G}}$ . This is the  $\beta$  function of the gravitational coupling constant when we choose the cosmological constant operator as the standard of the scale.

Although the cosmological constant operator is multiplicatively renormalizable, we have found that the quantum corrections are  $O(1)$ . Hence it may be useful to define the renormalized cosmological constant operator including the quantum corrections which are as large as the naive tree action.

The functional form of the renormalized cosmological constant operator may be very generic. However it can be fixed by the following method. Here we adopt the strategy which

has been successful for two dimensional quantum gravity[9, 10]. We decompose the metric into the background metric  $\hat{g}_{\mu\nu}$  and the fluctuations around it. In quantum gravity physical observables should be background independent. The functional form of the cosmological constant operator in terms of the conformal mode  $\psi$  should be such that it satisfies this requirement.

Let the renormalized cosmological constant operator to be

$$\int e^{-\frac{D}{2}\bar{\phi}}\Lambda(\psi). \quad (46)$$

We assume that the renormalized cosmological constant operator is invariant under the volume preserving diffeomorphism just like the tree action. It is also reasonable to assume that it only depends on  $\psi$  and  $\sqrt{\hat{g}} = e^{-\frac{D}{2}\bar{\phi}}$ . We expect that the counter terms may depend on generic fields. We parametrize the cosmological constant operator as:

$$\Lambda(\psi) = \exp(\alpha\psi + \frac{1}{2}\beta\psi^2 \dots). \quad (47)$$

In order to determine  $\Lambda(\psi)$ , we impose the invariance under the gauge transformation eq. (7) on this operator. If the theory is invariant under the volume preserving diffeomorphism, the gauge transformation (7) holds if the theory is invariant under the the conformal transformation (26). For this reason, we only need to require the invariance under the conformal transformation eq. (26) in order to impose the gauge invariance (7).

The important quantum effect is the anomalous dimension of the operator  $\Lambda(\psi)$ . At the one loop order,  $\Lambda(\psi)$  changes under the transformation eq. (26) as:

$$\delta\Lambda(\psi) = \frac{G}{8\pi} \frac{\partial^2}{\partial\psi^2} \Lambda(\psi) \delta\bar{\phi}. \quad (48)$$

The variation due to the  $\psi$  field transformation as in eq. (26) is

$$\delta\Lambda(\psi) = (D-1)(a+2\epsilon b\psi) \frac{\partial}{\partial\psi} \Lambda(\psi) \delta\bar{\phi}. \quad (49)$$

The sum of the above must cancel the variation of  $\delta(\sqrt{\hat{g}})\Lambda = -\frac{D}{2}\delta\bar{\phi}(\sqrt{\hat{g}})\Lambda$ . The coefficients  $\alpha, \beta, \dots$  are determined in this way as:

$$\alpha = \frac{4\pi a}{G} \left( -1 \pm \sqrt{1 + \frac{G}{2\pi a^2}} \right), \beta = -\frac{\epsilon\alpha}{4a + G\alpha/\pi}, \dots \quad (50)$$

We choose the + sign out of the two possible branches in the above expression since it possesses the correct semiclassical limit.

From the tree action eq. (3), we read off the effective gravitational coupling as the multiplication factor in front of  $\tilde{R}$ :

$$\frac{1}{G} - \frac{\delta G}{G^2} = \frac{1}{G}(1 + a\psi + \epsilon b\psi^2). \quad (51)$$

On the other hand, the log of the cosmological constant operator is found to be:

$$\log(\Lambda) = \alpha\psi + \frac{1}{2}\beta\psi^2 \dots \quad (52)$$

Since the quantum fluctuation of  $\psi$  is  $O(\sqrt{G})$ , it is at most  $O(\sqrt{\epsilon})$  even around the short distance fixed point. There is a scaling window for  $\psi < 1/\sqrt{\epsilon}$  since the nonlinear terms of  $\psi$  in the above expressions can be neglected then. From these reasonings, the  $\beta$  function is found to be

$$\frac{dG}{d\log(\Lambda)} = \frac{aG}{\alpha}. \quad (53)$$

At the short distance fixed point,  $\alpha \rightarrow \sqrt{8\pi/G}$ . Hence the above  $\beta$  function approaches  $aG^2/\sqrt{8\pi G}$  which agrees with our result in the first part of this section.

The relationship between the bare cosmological constant operator (36) and the renormalized one (46) may be understood as follows. We conjecture that the bare cosmological constant operator which is obtained after adding all necessary counter terms to the renormalized operator is manifestly invariant under the diffeomorphism. The precise relationship at the one loop level is:

$$\int d^D x \sqrt{g} = \int \exp\left(-\frac{G}{4\pi\epsilon} \frac{\partial^2}{\partial\psi^2}\right) \Lambda(\psi). \quad (54)$$

From this equation the renormalization group equation for  $\Lambda(\psi)$  follows:

$$\mu \frac{\partial}{\partial\mu} \Lambda(\psi) = \frac{G}{4\pi} \frac{\partial^2}{\partial\psi^2} \Lambda(\psi), \quad (55)$$

where we have used the leading renormalization group equation  $\mu \frac{d}{d\mu} G = \epsilon G$ .

The solution of this diffusion equation is

$$\Lambda(\psi) = \int_{-\infty}^{\infty} d\psi' \sqrt{\frac{\epsilon}{G}} \exp\left(-\frac{\pi\epsilon}{G}(\psi - \psi')^2\right) \Lambda_I(\psi'), \quad (56)$$

where  $\Lambda_I(\psi)$  is the initial condition of the renormalized operator. For the initial condition, we can assume the classical expression which is the same with the bare expression (36) at the weak coupling limit. This is because there should be no renormalization when the coupling is very weak. It naturally follows from our postulate (54).

We can evaluate this integral by the saddle point method again for small  $\epsilon$  limit:

$$\begin{aligned} \Lambda(\psi) &= \int_{-\infty}^{\infty} d\rho \exp\left(-\frac{16\pi a^2}{\epsilon G}(\rho - \frac{\epsilon}{4a}\psi)^2 + \frac{4}{\epsilon} \log(1 + \rho)\right) \\ &\sim \exp\left(\alpha\psi - \frac{1}{2} \frac{\epsilon\pi}{G} \psi^2\right), \end{aligned} \quad (57)$$

where  $\alpha$  is precisely the same coefficient with (50). The coefficient of  $\psi^2$  term is estimated at short distance where  $a$  is small which also agrees with (50). Therefore we have derived the functional form of the renormalized cosmological constant operator from the bare cosmological constant operator based on the postulate (54) at the one loop level.

The  $\beta$  function of  $G$  with respect to the cosmological constant we have studied in this section differs from the  $\beta$  function eq. (33) in the previous section which is related to the conformal anomaly. This situation reflects the inherent ambiguity of the definition of the  $\beta$  functions in quantum gravity. Namely they differ when different operator is chosen as the standard of the scale. However the both quantities vanish at the short distance fixed point where  $a = 0$ . From this sense the both definitions are consistent.

## 6 Conclusions and Discussions

In this paper we have further studied the renormalizability of quantum gravity near two dimensions. We thereby put the  $2 + \epsilon$  dimensional expansion of quantum gravity on a solid foundation. We have proven that all necessary counter terms can be supplied by the bare action which is invariant under the full diffeomorphism.

However the tree level action itself is not invariant under the general coordinate transformation. Only after adding the counter terms and thereby considering the bare action, we can recover the action which is invariant under the full diffeomorphism.

We have chosen the tree level action to possess the volume preserving diffeomorphism invariance. In order to recover the full diffeomorphism invariance, we need to require that the theory is independent of the background metric. This requirement has led us to search a theory which is conformally invariant with respect to the background metric. Obviously the Einstein action is such a theory and we conjecture that the requirement of the background independence leads us uniquely to the Einstein action as the bare action.

In our perturbative expansion, we need to introduce not only the gravitational coupling constant  $G$  but also another coupling constant  $a$ .  $G$  controls the dynamics of  $h_{\mu\nu}$  field while  $a$  controls the dynamics of the conformal mode.  $a^2$  is related to  $G$  since it is nothing but the  $\beta$  function of  $G$  modulo a factor of  $G$ . However  $a$  itself cannot be expanded in  $G$  since such an expansion is singular in  $\epsilon$ .

We have constructed a proof of the renormalizability of the theory to all orders in the perturbative expansion of  $G$ . In this expansion,  $a^2$  is also perturbatively determined in terms of  $G$ . Since  $G$  is  $O(\epsilon)$  at the short distance fixed point of the renormalization group,  $G$  may be regarded as a small expansion parameter as long as we regard  $\epsilon$  to be small. Our proof is base on the plausible assumption concerning the solution of eq. (25). This assumption is in accord with the investigations of the gauge theories[8, 12]. We hope that it can also be proven in the near future.

The renormalization of the cosmological constant operator is analogous. We have shown that the cosmological constant operator is multiplicatively renormalizable. The anomalous dimension of the cosmological constant operator is generically  $O(1)$  near two dimensions and we need to be more careful to calculate it. However it can be calculated for small  $\epsilon$  by the saddle point method. The exact two dimensional solutions can be understood in this way[3]. It may be useful to consider the renormalized cosmological constant operator which incorporates the quantum effect. We can require the background metric independence to

determine the renormalized cosmological constant operator in the analogous way with the two dimensional quantum gravity. The relationship between the bare and the renormalized cosmological constant operator is also considered. We need to add counter terms to the renormalized cosmological constant operator. We have postulated that the bare cosmological constant operator which is obtained in this way is manifestly invariant under the diffeomorphism.

In order to calculate the  $O(\epsilon)$  corrections, we need to perform the full two loop calculations. The partial results have been reported in [6]. We believe that we have made it clear how such a calculation can be completed.

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